Polarization Transformation in Twisted Anisotropic Media

R. E. VAN DOEREN

Abstract—**Polarization transformation of plane waves propagating in twisted anisotropic media is studied theoretically and numerically. It is shown that rotation of linear polarization is effected by such a medium when the anisotropy is of the order of 2 to 1 and twist rates commensurate with the relative value of the dielectric constants of the medium are used (less than $15°/\lambda_0$ for low dielectric constants and up to $90°/\lambda_0$ for dielectrics in the vicinity of 1000).**

I. Introduction

THE BEHAVIOR of an electromagnetic field in the presence of matter depends greatly on the spatial orientation, i.e., polarization, of the field vectors and the variation of the polarization as a function of time. For example, widely variant effects may be expected for different polarizations of a plane wave incident at Brewster's angle [1] on a plane interface between two dielectric media. Consideration of such effects is commonplace for the antenna-radome engineer. The study of polarization transformation phenomena, therefore, is of interest both academically and practically. This paper will emphasize the special case of rotation of linear polarization.

Certain media, called optically active, rotate the plane of linearly polarized light traversing the media. Optically active media have been discussed by many authors [2]–[5]. Fresnel demonstrated theoretically and experimentally that optical activity could be explained by the existence of different phase velocities for the right-circularly polarized (RCP) and left-circularly polarized (LCP) components of a linearly polarized wave; this is called circular birefringence.

According to Ditchburn [6] and Landau and Lifshitz [7], the tensor permittivity shown below gives the property of circular birefringence and, thus, perfect rotation of linear polarization.

$$[\epsilon] = \begin{bmatrix} \epsilon & j\gamma & 0 \\ -j\gamma & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix}.$$

Perhaps the best-known medium that exhibits optical rotatory power is crystalline quartz. The activity of quartz was first noted in 1811 by Arago [8] and was observed to occur when light propagated along the optical axis of the crystal. The quartz crystal possesses a helical arrangement of the silica molecules along its optical axis [9]. It was this twisted microscopic medium that initiated interest in theoretical considerations of the twisted macroscopic medium that is the subject of this paper.

Realization of such a twisted anisotropic medium might be made using a ferroelectric medium subjected to a twisted biasing field. Another twistable medium might be realized in the manner described by Collin [11], using alternating flexible layers of high and low dielectric materials. This particular medium (untwisted) is shown in Fig. 1.

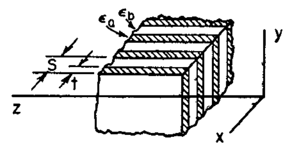

Fig. 1. Layered anisotropic medium.

For the medium in Fig. 1, static formulations may be used to approximate the effective dielectric constants for the electric field normal ($\epsilon_\perp$) and parallel ($\epsilon_\parallel$) to the layer boundaries if $s \ll \lambda$. Such a formulation and some calculated examples follow:

$$\epsilon_\perp = \left[\frac{\epsilon_b - \epsilon_a}{\epsilon_b \epsilon_a}\frac{t}{s} + \frac{1}{\epsilon_b}\right]^{-1}$$

$$\epsilon_\parallel = \epsilon_b + (\epsilon_a - \epsilon_b)\frac{t}{s}.$$

TABLE I
Dielectric Constants for a Layered Medium

ϵ_a	ϵ_β	t/s	$\epsilon_\parallel$	$\epsilon_\perp$
4.0	1.0	0.5	2.5	1.6
6.0	1.0	0.6	4.0	2.0
8.0	1.5	0.4	4.1	2.2
10.0	1.0	0.8	8.2	3.6
10.0	4.0	0.5	7.0	5.7

II. Theory of Propagation

Analysis of a TEM wave propagating in a twisted anisotropic medium will be performed. Time dependence of $e^{j\omega t}$ and lossless, linear media with magnetic permeability equal to that of free space will be assumed. The

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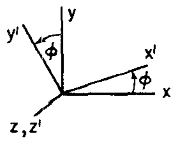
The author is with the Antenna Laboratory, Department of Electrical Engineering, The Ohio State University, Columbus, Ohio.

tensor permittivity $[\epsilon]$ will be taken as having only diagonal components and will be defined in terms of the twisted (primed) axes shown in Fig. 2.



$$[\epsilon] = \epsilon_0 \begin{bmatrix} \epsilon_{x'} & 0 & 0 \\ 0 & \epsilon_{y'} & 0 \\ 0 & 0 & \epsilon_{z'} \end{bmatrix}$$

Fig. 2. Twisted coordinate system.

The permittivity of free space is ϵ_0. The dimensionless quantities, therefore, are $\epsilon_{x'}$, $\epsilon_{y'}$, and $\epsilon_{z'}$, and, in this paper, are allowed to be functions of z.

The coordinate rotation equations for the twisted medium shown are

$$\begin{cases} \hat{x} = \hat{x}' \cos\phi - \hat{y}' \sin\phi \\ \hat{y} = \hat{y}' \cos\phi + \hat{x}' \sin\phi \\ \hat{z} = \hat{z}' \end{cases} \tag{1}$$

$$\begin{cases} \hat{x}' = \hat{x} \cos\phi + \hat{y} \sin\phi \\ \hat{y}' = \hat{y} \cos\phi - \hat{x} \sin\phi \\ \hat{z}' = \hat{z} \end{cases} \tag{2}$$

$$\begin{cases} E_{x'} = E_x \cos\phi + E_y \sin\phi \\ E_{y'} = E_y \cos\phi - E_x \sin\phi \\ E_{z'} = E_z. \end{cases} \tag{3}$$

The twist angle of the medium is ϕ and is taken to be a continuous function of z, $\phi = \phi(z)$.

The fundamental constitutive equation follows:

$$[D] = \begin{bmatrix} D_{x'} \\ D_{y'} \\ D_{z'} \end{bmatrix} = \epsilon_0 \begin{bmatrix} \epsilon_{x'} & 0 & 0 \\ 0 & \epsilon_{y'} & 0 \\ 0 & 0 & \epsilon_{z'} \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{bmatrix}. \tag{4}$$

Analysis of the polarization transformation properties of this medium will begin with Maxwell's equations.

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} = -j\omega\mu_0 \boldsymbol{H} \tag{5}$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D}. \tag{6}$$

Taking the curl of (5) and using (6) gives

$$\nabla \times \nabla \times \boldsymbol{E} = -j\omega\mu_0 \nabla \times \boldsymbol{H} = \omega^2\mu_0 \boldsymbol{D}. \tag{7}$$

Let us assume that $\partial/\partial x = \partial/\partial y = 0$ and that $E_z = 0$. It follows that

$$\nabla \cdot \boldsymbol{E} = \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = 0 \tag{8}$$

and

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E} = \omega^2 \mu_0 \boldsymbol{D}. \tag{9}$$

From (4),

$$\boldsymbol{D} = \epsilon_0 [\hat{x}' \epsilon_{x'} E_{x'} + \hat{y}' \epsilon_{y'} E_{y'} + \hat{z}' \epsilon_{z'} E_{z'}]. \tag{10}$$

E_z was assumed zero, and using the coordinate rotation expressions in (1),

$$\boldsymbol{D} = \hat{x}\epsilon_0[\epsilon_{x'} E_{x'} \cos\phi - \epsilon_{y'} E_{y'} \sin\phi] + \hat{y}\epsilon_0[\epsilon_{x'} E_{x'} \sin\phi + \epsilon_{y'} E_{y'} \cos\phi]. \tag{11}$$

Now, using (3) in (11) and collecting terms,

$$\begin{aligned} \boldsymbol{D} = {} & \hat{x}\epsilon_0[(\epsilon_{x'} \cos^2\phi + \epsilon_{y'} \sin^2\phi) E_x \\ & + (\epsilon_{x'} - \epsilon_{y'}) \sin\phi \cos\phi \, E_y] \\ & + \hat{y}\epsilon_0[(\epsilon_{x'} \sin^2\phi + \epsilon_{y'} \cos^2\phi) E_y \\ & + (\epsilon_{x'} - \epsilon_{y'}) \sin\phi \cos\phi E_x]. \end{aligned} \tag{12}$$

Equation (12) is substituted into (9), and equation of the $\hat{x}$ and $\hat{y}$ components gives

$$\frac{d^2 E_x}{dz^2} = -\omega^2 \mu_0 \epsilon_0 \left[(\epsilon_{x'} \cos^2\phi + \epsilon_{y'} \sin^2\phi) E_x + \frac{\epsilon_{x'} - \epsilon_{y'}}{2} \sin 2\phi E_y \right] \tag{13}$$

and

$$\frac{d^2 E_y}{dz^2} = -\omega^2 \mu_0 \epsilon_0 \left[(\epsilon_{x'} \sin^2\phi + \epsilon_{y'} \cos^2\phi) E_y + \frac{\epsilon_{x'} - \epsilon_{y'}}{2} \sin 2\phi E_x \right]. \tag{14}$$

It is convenient to set $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and to let $\delta = (\epsilon_{x'} - \epsilon_{y'})$. Equations (13) and (14) may be further condensed by use of

$$\begin{cases} \epsilon_{x'} \cos^2\phi + \epsilon_{y'} \sin^2\phi = \epsilon_{y'} + \delta \cos^2\phi \\ \epsilon_{x'} \sin^2\phi + \epsilon_{y'} \cos^2\phi = \epsilon_{y'} + \delta \sin^2\phi \end{cases} \tag{15}$$

and

$$\begin{cases} \sin^2\phi = \dfrac{1 - \cos 2\phi}{2} \\ \cos^2\phi = \dfrac{1 + \cos 2\phi}{2}. \end{cases} \tag{16}$$

Equations (13)and (14) become

$$\frac{d^2 E_x}{dz^2} = E_x'' = -k_0^2 \left[\left\{ \epsilon_{y'} + \frac{\delta}{2}(1 + \cos 2\phi) \right\} E_x + \frac{\delta}{2} \sin 2\phi E_y \right] \tag{17}$$

and

$$\frac{d^2 E_y}{dz^2} = E_y'' = -k_0^2 \left[\left\{ \epsilon_{y'} + \frac{\delta}{2}(1 - \cos 2\phi) \right\} E_y + \frac{\delta}{2} \sin 2\phi E_x \right]. \tag{18}$$

The magnetic field is determined by application of (5).

$$\boldsymbol{H} = \frac{j}{\omega\mu_0} \left[-\hat{x} \frac{\partial E_y}{\partial z} + \hat{y} \frac{\partial E_x}{\partial z} \right]. \tag{19}$$

III. Numerical Analysis

A. Conditions for the Analysis

The polarization transformation characteristics of the twisted media chosen for study were analyzed numerically for the situation shown in Fig. 3. Medium I is characterized by $\epsilon_{x'}$, $\epsilon_{y'}$, $\epsilon_{z'}$, and $\phi=\phi(z)$ with $\phi(o)=0$. Medium II is characterized by $\epsilon_{x'}$, $\epsilon_{y'}$, $\epsilon_{z'}$, and $\phi=0$.

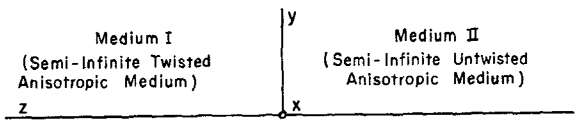


Fig. 3. Media for analysis.

In order to obviate the difficulties in satisfying the boundary conditions at the incident surface of a panel of the twisted medium, an inverse solution as used by Richmond [12] has been selected. One may specify a plane TEM wave traveling in the negative z direction at $z=0$ (the exit surface) and then use the exact differential equations to determine, via some numerical technique, the fields inside the medium corresponding to such an exit wave.

As pointed out in the Introduction, this study emphasizes the transformation of linear polarization. Accordingly, the exit wave was chosen to be linearly polarized and, additionally, was assigned a relative phase of zero. $\boldsymbol{E}_0$ is defined as the field at the exit surface where $\boldsymbol{E}_0=\hat{x}E_{x0}+\hat{y}E_{y0}$ and where $E_{x0}=E_0\cos\theta$, $E_{y0}=E_0\sin\theta$, and theta (θ) is the polarization angle of the exiting wave measured positively from OX toward OY.

B. Numerical Technique

If the field about the point $z=nh$ is expanded in a Taylor series,

$$E_{n+1}=E_n+hE_n'+\frac{h^2}{2}E_n''+\frac{h^3}{6}E_n'''+\frac{h^4}{24}E_n''''+\cdots \tag{20}$$

and

$$E_{n-1}=E_n-hE_n'+\frac{h^2}{2}E_n''-\frac{h^3}{6}E_n'''+\frac{h^4}{24}E_n''''+\cdots. \tag{21}$$

Addition of (20) and (21) gives, if the fourth-degree and higher terms are neglected,

$$E_{n+1}=h^2E_n''+2E_n-E_{n-1}. \tag{22}$$

Equation (22) is valid providing E is analytic in the interval $(n-1)h\leq z\leq(n+1)h$. (See also *Errors*.)

Thus, the field at $z=(n+1)h$ may be determined from the fields at $z=nh$ and $z=(n-1)h$ and from the second derivative of the field evaluated at $z=nh$. Equation (22) allows step-by-step solution for the fields throughout the medium. The increment in z between calculation points is, of course, h.

Initially, the value of the field at two points, (n) and ($n-1$), must be known. The field for $z\leq 0$ is given in the well-known exponential form

$$\boldsymbol{E}=\hat{x}E_{x0}e^{jk_xz}+\hat{y}E_{y0}e^{jk_yz}$$

where

$$k_x=k_0\sqrt{\epsilon_{x'}}\qquad k_y=k_0\sqrt{\epsilon_{y'}}.$$

Therefore, one may define the fields at $z=0$, calculate them at $z=-h$, and thus have two starting points.

For the sake of simplifying the notation of this problem, the subscripts 3, 2, and 1 will be used in place of ($n+1$), (n), and ($n-1$), respectively. Writing the difference equation (22) for E_x and E_y, and using the differential equations (17) and (18),

$$E_{x3}=\left\{2-(k_0h)^2\left[\epsilon_{y'}+\frac{\delta}{2}(1+\cos 2\phi_2)\right]\right\}E_{x2}-E_{x1}-(k_0h)^2\frac{\delta}{2}\sin 2\phi_2\Big)E_{y2} \tag{23}$$

and

$$E_{y3}=\left\{2-(k_0h)^2\left[\epsilon_{y'}+\frac{\delta}{2}(1-\cos 2\phi_2)\right]\right\}E_{y2}-E_{y1}-(k_0h)^2\left(\frac{\delta}{2}\sin 2\phi_2\right)E_{x2}. \tag{24}$$

For computational convenience, (23) and (24) are broken down into their real and imaginary parts using the following convention: $E_{x3}=U_{x3}+jV_{x3}$, $E_{y3}=U_{y3}+jV_{y3}$, etc.

The axial ratio and polarization angle ψ are calculated as follows:

$$\text{Axial Ratio}=\frac{|R_3|+|L_3|}{|R_3|-|L_3|}. \tag{25}$$

The polarization angle ψ is the angle from the x axis to the major axis of the polarization ellipse and is given by

$$\tan 2\psi=\frac{2(U_{x3}U_{y3}+V_{x3}V_{y3})}{|E_x|^2-|E_y|^2}. \tag{26}$$

$|R_3|$ and $|L_3|$ represent the magnitudes of the right-circularly and left-circularly polarized components, respectively, and are given by

$$R_3=(U_{x3}-V_{y3})+(V_{x3}+U_{y3})$$
$$L_3=(U_{x3}+V_{y3})+(V_{x3}-U_{y3}).$$

C. Errors

The step size h figures strongly in the magnitude of the error. The error introduced in neglecting the fourth-degree term may be taken as being of the same order of magnitude as that term itself. Examination of this term in light of a defined maximum permissible error per step will aid in placing an upper bound on the size of h.

Differentiation of (17) yields for the fourth derivative (taking $\epsilon_{y'}$ and δ to be constants)

$$\begin{aligned} E_x'''' = &- k_0^2\left\{\left[\epsilon_{y'} + \frac{\delta}{2}(1+\cos 2\phi)\right]E_x'' \right.\\ &- (\delta \sin 2\phi)(\phi')E_{x'} - (2\delta\cos 2\phi)(\phi')^2E_x \\ &- (\delta\sin 2\phi)(\phi'')E_x - (\delta \sin 2\phi)(\phi')E_x' \\ &- (2\delta\sin 2\phi)(\phi')^2E_y + (\delta\cos 2\phi)(\phi'')E_y \\ &+ (\delta\cos 2\phi)(\phi')E_y' + (\delta \cos 2\phi)(\phi')E_y' \\ &\left. + \left(\frac{\delta}{2}\sin 2\phi\right)E_y''\right\}. \end{aligned} \tag{27}$$

E_x'' is, of course, given by (17),

$$E_x'' = - k_0^2\left\{\left[\epsilon_{y'} + \frac{\delta}{2}(1+\cos 2\phi)\right]E_x + \frac{\delta}{2}\sin 2\phi E_y\right\}.$$

The dominant term of (27) is the first, i.e.,

$$E_x'''' \doteq - k_0^4\left\{\left[\epsilon_{y'} + \frac{\delta}{2}(1+\cos 2\phi)\right]^2 E_x + \left(\frac{\delta}{2}\sin 2\phi\right)\left[\epsilon_{y'} + \frac{\delta}{2}(1+\cos 2\phi)\right]E_y\right\}. \tag{28}$$

Let us examine the case for which the first term in (28) is dominant (ϕ and/or E_y small). Then,

$$E_x'''' \doteq - k_0^4\left[\epsilon_{y'} + \frac{\delta}{2}(1+\cos 2\phi)\right]^2 E_x. \tag{29}$$

For a medium with the greater of its two dielectric constants on the order of 10.0 ($\epsilon_{y'}+\delta \le 10$), assignment of the value 10.0 to the bracketed term in (29) will provide a realistic evaluation of an upper bound for h for a prescribed maximum error per step. Similarly, for media with dielectric constants on the order of 1000 (characteristic of the ferroelectric titanates), assignment of 1000 as the value of the bracketed term in (29) will provide an upper bound for h.

Let the maximum error per step be $\epsilon = 10^{-4}E_x$, or 0.01 percent per step. It is desired that the neglected term always be less than ϵ.

The numeric process of defining an upper bound for h proceeds as follows for the lower dielectric constant media:

$$\frac{h^4}{12}k_0^4(10.0)^2E_x \le 10^{-4}E_x = \epsilon$$

$$(k_0h) \le 0.059$$

$$\frac{h}{\lambda_0} \le 0.01.$$

A similar process for the high dielectric constant media ($\epsilon \approx 1000$) yields for the same error,

$$\frac{h}{\lambda_0} \le 0.001.$$

Accordingly, for the respective media, step sizes of $h/\lambda_0 = 0.01$ and $h/\lambda_0 = 0.001$ were used. Test computations were made on the lower dielectric constant media ($\epsilon_{\max} \le 10$) using step sizes of 0.001 in addition to the prescribed steps of 0.01; the results agreed within one part in the third decimal place for the field magnitudes. It was concluded that the step sizes as determined were satisfactorily small.

Use of the difference equation for $z \ge 0$ presents no difficulties; however, application of the difference equation to calculate the fields at $z=h$ in the interval $-h \le z \le h$ requires further consideration since the derivatives of the twist angle are discontinuous at $z=0$. In this situation, the fields may be expanded by Taylor's formula with the Lagrange form of the remainder [13], and (22) can be derived as before, providing the remainder terms may be neglected. For a step size of $h/\lambda_0 = 0.01$, an error on the order of 0.3 percent is expected in the calculation of the fields at $z=h$. Although higher than that expected inside the medium, such an error is considered acceptable.

IV. Numerical Results

Any anisotropic medium of the type studied would be expected to transform a nonprincipally polarized (E vector not directed perpendicular or parallel to the axis of greater dielectric constant) linearly polarized wave by virtue of the different phase velocities of its principally polarized components. Thus, one would expect a linearly polarized (at the exit surface) wave of the above type to result in elliptically, then circularly, then elliptically, then linearly polarized, etc., fields as one moved back through the medium. This type of polarization transformation per se is considered of little interest here. Since superposition of principally polarized components at the exit surface can synthesize any polarization of interest, attention has been devoted solely to the principal polarizations.

Media with low anisotropy ratios, e.g., $\epsilon_{y'}=4.0$ and $\epsilon_{x'}=4.5$, were found uninteresting as polarization transformers with either slow or rapid ($3°/\lambda_0$ to $360°/\lambda_0$) twist rates.

Several media with anisotropy ratios of 2 to 1 were analyzed for varying twist rates, and these proved very interesting. The numerical results for these media are shown graphically for the following cases:

a) $\epsilon_{y'} = 4.0 \qquad \epsilon_{x'} = 8.0.$

$\phi = (3.0°/\lambda_0)z, \quad (6.0°/\lambda_0)z, \quad (9.0°/\lambda_0)z, \quad (12.0°/\lambda_0)z, \text{ and } (15.0°/\lambda_0)z.$

Both parallel and perpendicular exit polarizations are shown in Figs. 4, 5, 6, and 7.

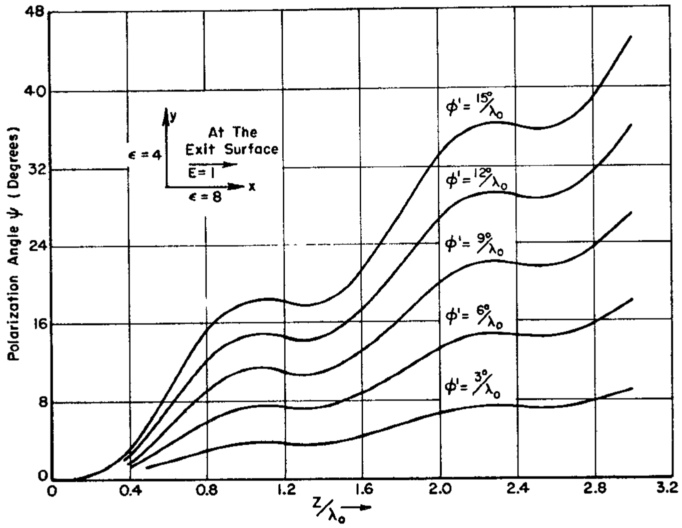


Fig. 4. Polarization angle vs. depth into medium.

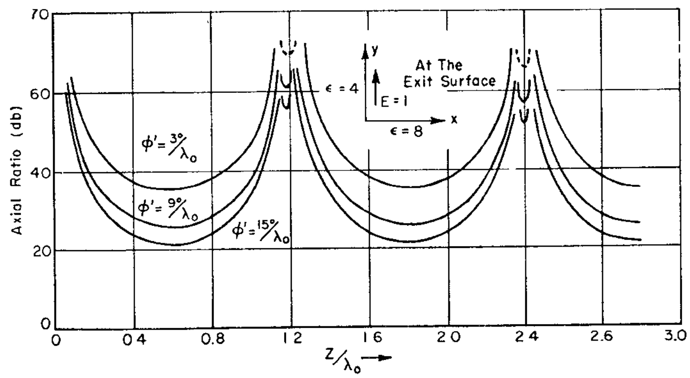


Fig. 7. Axial ratio vs. depth into medium.

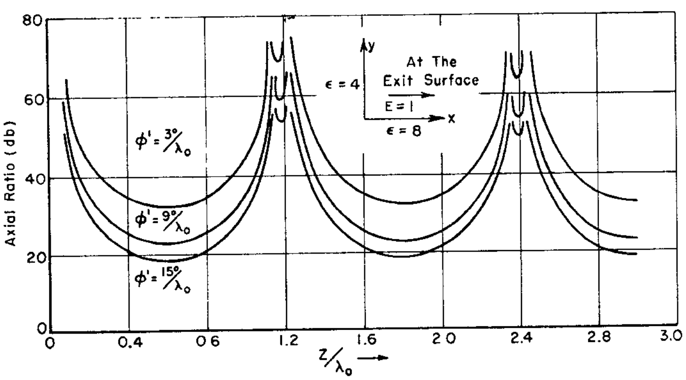


Fig. 5. Axial ratio vs. depth into medium.

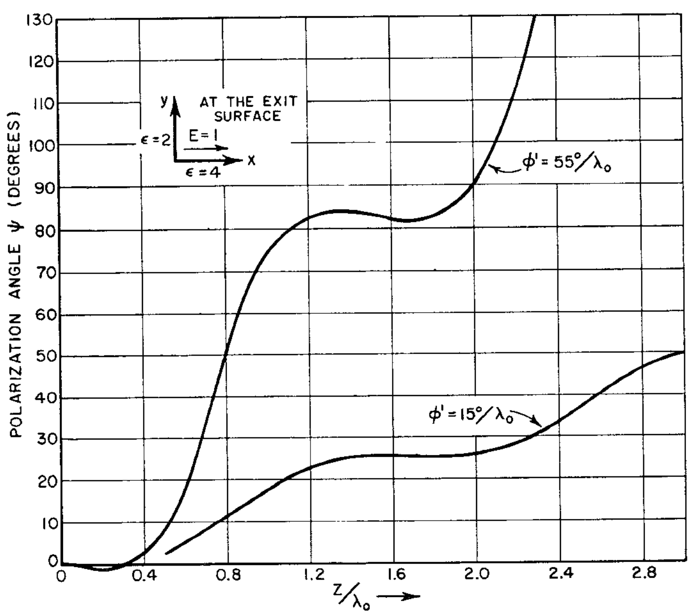


Fig. 8. Polarization angle vs. depth into medium.

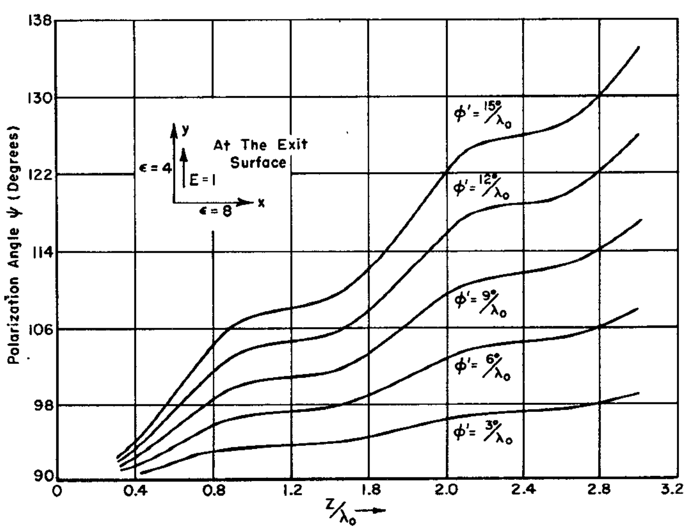


Fig. 6. Polarization angle vs. depth into medium.

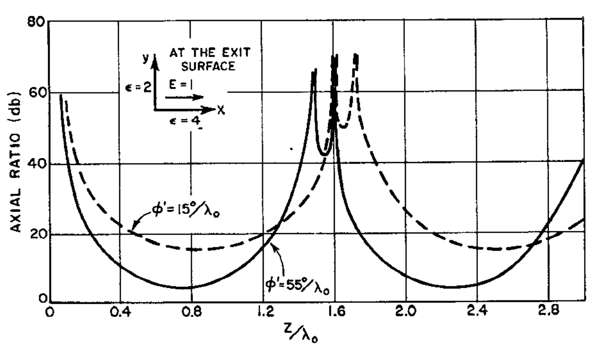


Fig. 9. Axial ratio vs. depth into medium.

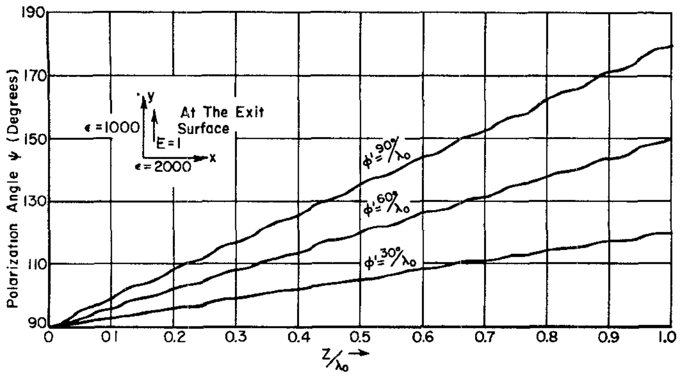


Fig. 10. Polarization angle vs. depth into medium.

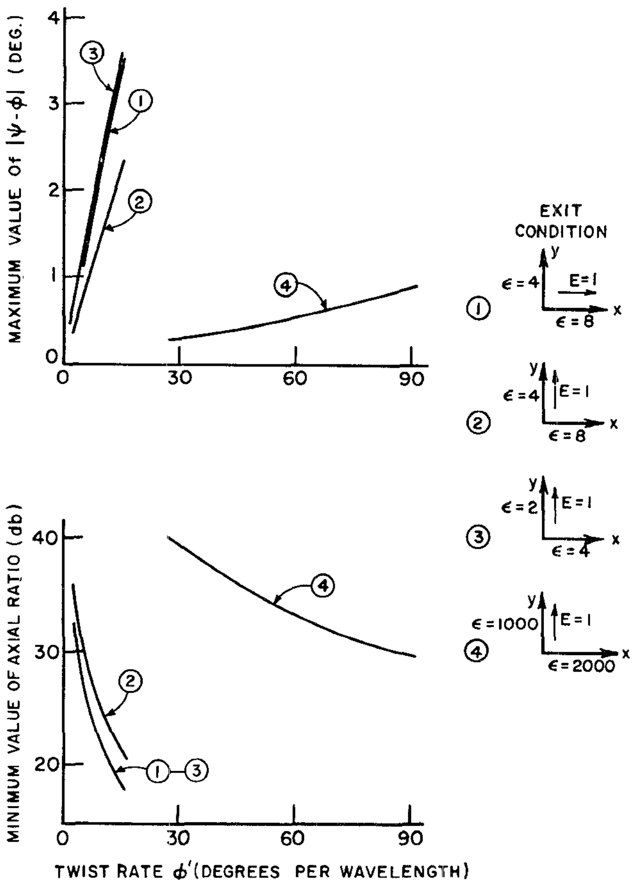


Fig. 11. Polarization angle deviation vs. twist rate.

Fig. 12. Axial ratio minimum vs. twist rate.

b) $\epsilon_{y'} = 2.0 \qquad \epsilon_{x'} = 4.0.$

Parallel exit polarization for $\phi = (15.0°/\lambda_0)z$, $(55.0°/\lambda_0)z$ is shown in Figs. 8 and 9.

c) $\epsilon_{y'} = 1000 \qquad \epsilon_{x'} = 2000.$

$$\phi = (30°/\lambda_0)z, \qquad (60°/\lambda_0)z, \quad \text{and} \quad (90°/\lambda_0)z.$$

Perpendicular exit polarization only is shown in Fig. 10.

The results are presented in the form of polarization angle ψ of the major axis of the polarization ellipse vs. depth into the twisted medium in free space wavelengths with the twist rate as parameter. For the axial ratio curves, the labeled rates of twist apply in the same vertical order for the unlabeled portions. The sign of the axial ratio changed at the points of the asymptotes, indicating a change in sense of the elliptically polarized wave, and the sense of elliptical polarization was opposite in the broad portions of the axial ratio curves for the two exit polarizations.

Figures 11 and 12 summarize the performance of the media studied. Figure 11 shows the maximum deviation of the polarization angle from the twist angle, and Fig. 12 shows the minimum value of the axial ratio, both vs. the twist rate.

V. Conclusions

The twisted anisotropic media studied (anistropy ratio 2 to 1) show marked polarization rotation. Least deviation of the polarization angle from the twist angle and higher axial ratios were obtained for the perpendicular exit polarization. For $\epsilon_{y'} = 4.0$, $\epsilon_{x'} = 8.0$ and $\epsilon_{y'} = 2.0$, $\epsilon_{x'} = 4.0$ media, twist rates up to 15° per free space wavelength were imposed, and the axial ratio minimum was greater than 15 dB and the polarization angle remained within four degrees of the twist angle. Twist rates of 90° per free space wavelength were imposed on the $\epsilon_{y'} = 1000$, $\epsilon_{x'} = 2000$ media (constants typical of the ferroelectric titanates), and for these relatively rapid twists, the axial ratio minimum was greater than 28 dB and the polarization angle deviated from the twist angle by less than one degree. These media, therefore, give highly respectable rotation of the plane of linear polarization.

The parallel exit polarization does not give as good linear polarization rotation as perpendicular; however, a constant angle of rotation for a considerable range of depths into the media can be observed for parallel polarization. Figures 8 and 9 show that a polarization angle of $82.7° \pm 1.5°$ from depths of 1.16 λ_0 to 1.82 λ_0 with axial ratios better than 15 dB can be obtained. This provides rather broadband, good quality fixed rotation of linear polarization.

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